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THE NORMAL FORM OF PERTURBATIONS OF NON-LINEAR OSCILLATORY SYSTEMS†

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The normal form of perturbations of a non-linear oscillatory system is defined. The system itself, called the generating system, is arbitrary in form. An algorithm is developed that enables one, without touching the generating system, to reduce a perturbation to normal form. The Campbell–Hausdorff series, the properties of the ring of asymptotics forms and the explicit solution of the homological equation are used to derive a one-dimensional recurrence formula of arbitrary approximation. © 2003 Elsevier Science Ltd. All rights reserved.

Poincaré's normal form methods [1, 2] consists in establishing the simplest form to which a system of ordinary differential equations can be reduced in the neighbourhood of an equilibrium position, and in presenting an algorithm for this reduction. It is assumed when doing so that the linear part of the system has already been reduced to the simplest form, that is, to Jordan form, after which one tries, by transformations not affecting the linear part, to eliminate all non-resonant terms. Since there are far less resonant terms, the initial system is simplified considerably, as is indeed the main goal of Poincaré's method.

At the same time, a differential system in normal form possesses the extremely important property that the vector fields of the linear and non-linear parts commute. Consequently, the non-linear part of the system generates the symmetry of its linear part and hence that of the entire system. This property is well known [3] and has been effectively used in practice. Being a geometrical property, it is, in particular, independent of the specific variables in which the vector fields are described. Hence it follows that preliminary reduction of the linear part to normal Jordan form is not necessary if the aim of the transformations is to establish the aforementioned geometrical fact. In Poincaré's method, such reduction is involved only in the actual procedure of constructing the normal form.

The method developed below enables the aim in question to be realized irrespective of the specific form in which the linear part of the oscillatory system is presented. In addition, it is not assumed that one of the parts of the system must necessarily be linear. The algorithm presented is associated with a single condition – that the commutator of the perturbation and the generating part of the system vanish.

For differential systems in Hamiltonian form, ideas similar to those described here have already been applied [4, 5].

Let us consider a system of ordinary differential equations in normal Cauchy form, as follows:

$$dx/dt = X(x, \varepsilon) = X_0(x) + \varepsilon X_*(x, \varepsilon) \tag{1}$$

where x is a real vector of arbitrary dimensionality, and the right-hand side is sufficiently smooth in the domain of definition. The real scalar parameter ε is assumed to be small. System (1) has the form of a perturbed system with generating part $dx/dt = X_0(x)$, whose general solution $x = \varphi(t, c)$, where $x(0) = c$, is assumed to be known. The vector field $\varepsilon X_*(x, \varepsilon)$ is called the perturbation. As usual, we are interested in the behaviour of the complete perturbed system in a small neighbourhood of the generating system.

If $X_0(x)$ is a linear function, system (1) corresponds to the system considered the theory of Poincaré's normal form after a small scale of measurement has been introduced for the dependent variable: $x \to \varepsilon x$. System (1) is more general than in Poincaré's method, because its generating part is non-linear, and moreover the small parameter may also be present without preliminary scaling. However, unlike Poincaré's method, we shall consider only oscillating generating systems, that is, we shall assume that the general solution $x = \varphi(t, c)$ is a conditionally periodic function of time [1].

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Definition. We shall say that the perturbation $\varepsilon X_*(x, \varepsilon)$ in system (1) is in normal form if the Poisson bracket (commutator) of the vector fields $X_0(x)$ and $X_*(x, \varepsilon)$ vanishes:

$$[X_0(x),\ X_*(x, \varepsilon)] = 0 \tag{2}$$

Problem. Suppose the perturbation in system (1) does not have normal form. It is required to find a change of variables $x \to y$ which, without changing the generating system, will reduce the perturbation to normal form.

In other words, we are looking for a nearly identical transformation $x \to y$ which will reduce system (1) to the form

$$dy/dt = Y_0(y, \varepsilon) = X_0(y) + \varepsilon Y_*(y, \varepsilon) \tag{3}$$

in which $[X_0(y), Y_*(y, \varepsilon)] = 0$.

In this form, the generating part of system (3) induces the symmetry group of the complete system; for this reason, the order of the latter may be reduced. In addition, the principle of separation of motions is realizable in system (3). This means that if one knows some particular solution of the added system

$$dy/dt = \varepsilon Y_*(y, \varepsilon)$$

in the form $y = \xi(t)$, then there is a corresponding solution of the complete system (3) which is obtained by simply substituting the solution $y = \xi(t)$ for the arbitrary constants c into the general solution of the generating system: $y = \varphi(t, \xi(t))$.

The tool we shall use to solve the problem posed above is the theory of local Lie groups [6, 7]. With that in mind, the phase flow induced by system (1) will be considered as a one-parameter Lie group with operator

$$A = X(x, \varepsilon)\frac{\partial}{\partial x} \tag{4}$$

which, in keeping with the form of system (1), will also be expressed as the sum of a generating part and a perturbation: $A = A_0 + A_*$. Condition (2), in terms of vector fields, is equivalent to the similar condition in terms of operators: $[A_0, A_*] = 0$.

The change of variables $x \to y$ will also be sought as a one-parameter Lie group

$$y = \exp(\tau U)x, \quad U = Z(x, \varepsilon)\frac{\partial}{\partial x} \tag{5}$$

where τ is the group parameter and U is its operator. In other words, the change of variables (5) is the general solution of the differential system

$$dy/d\tau = Z(y, \varepsilon), \quad y(0) = x$$

The group with operator U transforms the operator A into an operator B. The relation between these three operators is defined by a Campbell–Hausdorff series

$$B = A + \tau[A, U] + \frac{\tau^2}{2!}[[A, U], U] + \ldots, \tag{6}$$

or, equivalently

$$Y(y, \varepsilon) = X(y, \varepsilon) + \tau[X, Z] + \frac{\tau^2}{2!}[[X, Z], Z] + \ldots$$

We define a kth order asymptotic form of the operator U with respect to the small parameter ε to be any operator U_k which differs from the exact operator by quantities of order ε^{k+1} and higher: $U_k = U + O(\varepsilon^{k+1})$. Similar notation will be used for the other operators, A_k and B_k.

We recall the main properties of the ring of asymptotic forms.

1. Addition: $(A' + A'')_k = A'_k + A''_k$. The zero element of the ring of kth order asymptotic forms is any operator of order higher than k: $0_k = O(\varepsilon^{k+1})$.

2. Multiplication: $(A' \cdot A'')_k = A'_k \cdot A''_k$. The unit element is $1_k = 1 + O(\varepsilon^{k+1})$ (the ring of asymptotic forms is unitary).
3. Displacement along the scale of orders: $(\varepsilon^s \cdot A)_k = \varepsilon^s A_{k-s}$ $(k - s \geqslant 0)$.

In what follows, we shall consider the problem of successively finding the asymptotic forms of the unknown operators B and U, starting from the lowest-order ones.

Using the properties listed above and also identifying τ with ε, we can rewrite the infinite Campbell–Hausdorff series (6) for the asymptotic forms of vector fields beginning with the lowest order as a sequence of finite series

$$B_0 = A_0, \quad B_1 = A_1 + \varepsilon[A_0, U_0], \quad B_2 = A_2 + \varepsilon[A_1, U_1] + \frac{\varepsilon^2}{2!}[[A_0, U_0], U_0], \ldots$$

$$\ldots, B_k = A_k + \varepsilon[A_{k-1}, U_{k-1}] + \sum_{i=2}^{k} \frac{\varepsilon^i}{i!} \underbrace{[\ldots[}_{i \text{ times}} A_{k-i}, U_{k-i}], U_{k-i}], \ldots] \tag{7}$$

In the expression for B_k, consider the term $\varepsilon[A_{k-1}, U_{k-1}]$. Using the operator of displacement along the scale of orders, we can express this term as follows:

$$\varepsilon[A_{k-1}, U_{k-1}] = \varepsilon[A_{k-1} - A_0, U_{k-1}] + \varepsilon[A_0, U_{k-1}] = \varepsilon[A_{k-1} - A_0, U_{k-2}] + \varepsilon[A_0, U_{k-1}]$$

(we have used the fact that $A_{k-1} - A_0 = O(\varepsilon)$). Using this relation, we can rewrite the expression for B_k as an explicit one-dimensional recurrence formula

$$B_k = \varepsilon[A_0, U_{k-1}] + R_k, \quad k = 1, 2, \ldots \tag{8}$$

which enables us successively, beginning with $k = 1$, to determine all the asymptotic forms of the operators B and U up to any necessary order inclusive. Here R_k is an operator expressed only in terms of asymptotic forms of U of lower order than U_{k-1}

$$R_k = A_k + (1 - \delta_{1k})\varepsilon[A_{k-1} - A_0, U_{k-2}] + \sum_{i=2}^{k} \frac{\varepsilon^i}{i!} \underbrace{[\ldots[}_{i \text{ times}} A_{k-i}, U_{k-i}], U_{k-i}], \ldots] \tag{9}$$

$$(\delta_{11} = 1, \ \delta_{1k} = 0, \ k > 1)$$

The operator R_k, which has a central role to play in the recurrence scheme being formulated, will be called the resolvent of the scheme.

Let us consider the first approximation of the normal form

$$B_1 = \varepsilon[A_0, U_0] + A_1 \tag{10}$$

In this equation, known as the homological equation [8]. the unknowns are B_1 and U_0. To find B_1, consider Eq. (10) along trajectories of the generating system

$$dy/dt = X_0(y), \quad y(0) = c \Rightarrow y = \varphi(t, c) \tag{11}$$

This means that, assuming that all the operators in (10) are expressed in terms of the variable y, we must transform from this variable to the integration constant c according to formula (11).

The change of variables in the operator A_1 is accomplished as follows:

$$A_1 = X_1(y, \varepsilon)\frac{\partial}{\partial y} \Rightarrow \overline{A}_1 = \overline{X}_1(t, c, \varepsilon)\frac{\partial}{\partial c}: \quad \overline{A}_1 = [A_1 \varphi(-t, y)|_{y=\varphi(t,c)}]\frac{\partial}{\partial c}$$

The other operators in (10) are transformed similarly, but by assumption the operator B_1 commutes with the operator A_0 and consequently its form is unaffected by the change of variable: $\overline{B}_1 = B_1$, while the operator A_0, $\overline{U}_0$ along trajectories of the generating system equals the time derivative of the operator $\overline{U}_0$: $[A_0, \overline{U}_0] = d\overline{U}_0/dt$ $(\overline{A}_0 = A_0)$. As a result, Eq. (10), in terms of the variables c, becomes

$$B_1 = \varepsilon d\overline{U}_0/dt + \overline{A}_1 \tag{12}$$

Evaluating the time average of this equation, we obtain $B_1 = \langle \bar{A}_1 \rangle$, since B_1 is independent of time and the average of the derivative vanishes because the solution of the generating system is conditionally periodic. Substituting this expression for B_1 into formula (12), we obtain an equation for $\bar{U}_0$

$$\varepsilon \frac{d\bar{U}_0}{dt} = B_1 - \bar{A}_1 = -\tilde{\bar{A}}_1, \quad \tilde{\bar{A}}_1 = \tilde{\bar{X}}_1(t, c, \varepsilon)\frac{\partial}{\partial c}$$

where the operator $\tilde{\bar{A}}_1$ does not have an average value with respect to time along trajectories of the generating system. Integrating this equation, we obtain

$$\bar{U}_0 = -\frac{1}{\varepsilon}\int \tilde{\bar{A}}_1 dt$$

or, returning to the original variable y,

$$U_0 = \bar{U}_0 \varphi(t, c)\big|_{c=\varphi(-t,y)} \frac{\partial}{\partial y}$$

The two operations performed here – evaluating the average, which has made it possible to find a first approximation B_1 to the normal form, and solving the homological equation – may be combined in a single operator, by simply integrating Eq. (12) with respect to time along trajectories of the generating system

$$\int_0^t \bar{A}_1 dt = tB_1 + \varepsilon U_0 - \varepsilon \bar{U}_0 \tag{13}$$

Since $\bar{B}_1 = B_1$ and $\bar{U}_0|_{t=0} = U_0$, it follows from formula (13) that, integrating the known operator A_1 along the general solution of the generating system, we obtain the desired operator B_1 as the coefficient of t, and the other desired operator U_0 as the time-independent coefficient of ε.

To construct a second approximation of the normal form, we have to apply a change of variables similar to the previous one (11) in the next equation of system (7)

$$B_2 = \varepsilon[A_0, U_1] + A_2 + \varepsilon[A_1 - A_0, U_0] + \frac{\varepsilon^2}{2}[[A_0, U_0], U_0]$$

after which integration with respect to time yields

$$\int_0^t \bar{R}_2 dt = tB_2 + \varepsilon U_1 - \varepsilon \bar{U}_1$$

As in the first approximation, the coefficient of t will be the desired asymptotic form B_2 and the next asymptotic form U_1 will be the time-independent coefficient of ε. The operator R_2 is expressed by formula (9), in which the operator U_0 has already been found (in the preceding step).

We will now formulate the entire algorithm for constructing an arbitrary approximation to the normal form, assuming that all previous ones have already been constructed, i.e. it is required to find B_k and U_{k-1}, on the assumption that all lower-order asymptotic forms are known. The algorithm contains three main steps.

1. Use formula (9) to compute the resolvent R_k.
2. Transform to the integration constant c in R_k by the formula

$$\bar{R}_k = [R_k \varphi(-t, y)]_{y=\varphi(t,c)} \frac{\partial}{\partial c} \tag{14}$$

3. Find the normal form by averaging (14) with respect to time, which occurs explicitly in it.

$$B_k = \langle \bar{R}_k \rangle = \lim_{T\to\infty} \frac{1}{T}\int_0^T \bar{R}_k dt$$

4. In the variables c, evaluate the corresponding asymptotic form of the operator of the normalizing transformation operator

$$\overline{U}_{k-1} = -\frac{1}{\varepsilon}\int[\overline{R}_k - \langle\overline{R}_k\rangle]dt$$

5. Transform to the variable y

$$U_{k-1} = \overline{U}_{k-1}\varphi(t, c)\big|_{c=\varphi(-t,y)}\frac{\partial}{\partial y}$$

Here the algorithm ends; however, as already noted when the first approximation was constructed, steps 3–5 may be replaced by a single step:

6. Evaluate the integral with respect to t of $\overline{R}_k$. The unknowns B_k and U_{k-1} will then satisfy the relations

$$\int_0^t \overline{R}_k dt = tB_k + \varepsilon U_{k-1} - \varepsilon\overline{U}_{k-1} \tag{15}$$

if the separation of the terms occurring on the right-hand side is obvious and presents no difficulties.

The operator U_{k-1} we have found is expressed in terms of the space variables y. By formula (5), this same operator, written in terms of the space variables x, yields the relation between the variables x and y

$$y = \exp(\varepsilon U_{k-1})x \tag{16}$$

The same operator, written in terms of the variables y, defines the inverse transformation

$$x = \exp(-\varepsilon U_{k-1})y \tag{17}$$

Example. As an example, consider the Van der Pol equation

$$\ddot{x} + x = \varepsilon(1 - x^2)\dot{x}$$

which, in phase variables, may be written as the system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + \varepsilon(1 - x_1^2)x_2 \tag{18}$$

Consider the following problem: without changing the linear part of this system, in particular, without reducing it to Jordan form, find the normal form of the non-linear terms in the sense defined above, and also the change of variables required for this.

The vector field defined by the system is

$$X(x_1, x_2) = \{x_2, -x_1 + \varepsilon(1 - x_1^2)x_2\}$$

By (9), the resolvent R_1 needed to construct a first approximation is in this case

$$R_1 = A_1 = A = y_2\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial y_2} + \varepsilon(1 - y_1^2)y_2\frac{\partial}{\partial y_2}$$

The general solution of the generating system may be written as

$$y_1 = c_1\cos t + c_2\sin t, \quad y_2 = -c_1\sin t + c_2\cos t \tag{19}$$

By (14), the operator R_1 must be reduced to integration constants (c_1, c_2)

$$\overline{R}_1 = R_1(y_1\cos t - y_2\sin t)\frac{\partial}{\partial c_1} + R_1(y_1\sin t + y_2\cos t)\frac{\partial}{\partial c_2} \tag{20}$$

where the variables y in the components of the operator obtained must be replaced by the constants c obtained by inversion of formulae (19). The result is

$$\bar{R}_1 = r_1(c_1, c_2, t)\frac{\partial}{\partial c_1} + r_2(c_1, c_2, t)\frac{\partial}{\partial c_2}$$

where

$$r_j = (-1)^{j+1}c_{3-j} + \frac{\varepsilon}{8}c_j[4 - c_1^2 - c_2^2 - 4(1 - c_{3-j}^2)\cos 2t + (c_j^2 - 3c_{3-j}^2)\cos 4t] -$$

$$-\frac{\varepsilon}{8}c_{3-j}[2(2 + (-1)^{j+1}(c_1^2 - c_2^2))\sin 2t - (-1)^{j+1}(3c_j^2 - c_{3-j}^2)\sin 4t], \quad j = 1, 2$$

We now evaluate the integral with respect to t of the operator $\bar{R}_k$, implementing the third, last stage of the algorithm (formula (15))

$$\int_0^t \bar{R}_1 dt = t\left[f_1(y_1, y_2)\frac{\partial}{\partial y_1} + f_2(y_1, y_2)\frac{\partial}{\partial y_2}\right] - \varepsilon\left[g_1(y_1, y_2)\frac{\partial}{\partial y_1} + g_2(y_1, y_2)\frac{\partial}{\partial y_2}\right]$$

$$f_j = (-1)^{j+1}y_{3-j} + \frac{\varepsilon}{8}y_j(4 - y_1^2 - y_2^2), \quad j = 1, 2$$

$$g_1(y_1, y_2) = \frac{1}{32}y_2(8 + y_1^2 - 3y_2^2), \quad g_2(y_1, y_2) = \frac{1}{32}y_1(8 - 5y_1^2 + 7y_2^2)$$

Since the operator $\bar{U}_0$ is of no interest, we shall not write it down explicitly.

The expression obtained contains complete information on the normal form of the Van der Pol system (18) and the appropriate transformation. The normal form is determined by the coefficient of t

$$\dot{y}_1 = f_1(y_1, y_2), \quad \dot{y}_2 = f_2(y_1, y_2) \tag{21}$$

The time-independent coefficient of ε defines the transformation operator

$$U_0 = -\left[g_1(y_1, y_2)\frac{\partial}{\partial y_1} + g_2(y_1, y_2)\frac{\partial}{\partial y_2}\right]$$

By formulae (16) and (17), this operator determines the direct and inverse changes of variables relating systems (18) and (21). In particular, the direct transformation is

$$y_1 = x_1 - \varepsilon g_1(x_1, x_2), \quad y_2 = x_2 - \varepsilon g_2(x_1, x_2)$$

Remarks. 1. If computations using formula (9) lead to uncertain terms (the zero of the ring of asymptotic forms), they must be equated to zero in order to simplify the derivations.

2. If the generating system is linear and diagonal form, the algorithm yields the usual Poincaré normal form, but via much more economical manipulations than when the known algorithm is used. As already remarked, the invariant nature of our algorithm avoids the need to reduce the linear part to diagonal form. However, it is convenient to begin the construction of the higher-order approximations with a diagonal linear part, since it is more economical to work with exponential functions than with trigonometric functions.

If the generating system has the form characteristic for single- or multi-frequency systems of the Krylov–Bogolyubov method, the algorithm presented above yields the construction of the corresponding averaged systems to within any desired approximation [9].

3. If the eigenfrequencies of the linear generating system are close to some resonance condition, small denominators will appear; such cases should be converted to exact resonance by deleting small linear terms in the perturbation.

4. The need to compute quadratures in the case of the general position may lead to difficulties in using formula (15). However, if, as is usually assumed, the differential system under consideration is expressed in polynomial form, one has to integrate exponential or trigonometric harmonics, which is easily done.

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